

# ON THE EQUIVALENCE OF SOLUTIONS FOR A CLASS OF STOCHASTIC EVOLUTION EQUATIONS IN A BANACH SPACE

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**ABSTRACT.** We study a class of stochastic evolution equations in a Banach space  $E$  driven by cylindrical Wiener process. Three different concept of solutions: generalised strong, weak and mild are defined and the conditions under which they are equivalent are given. We apply this result to prove existence, uniqueness and continuity of weak solutions to stochastic delay equation with additive noise. We also consider two examples of these equations in non-reflexive Banach spaces: a stochastic transport equation with delay and a stochastic McKendrick equation with delay.

## 1. INTRODUCTION

Let  $E$  be a Banach space and let  $H$  denote a separable Hilbert space. Consider the stochastic evolution equation in  $E$ :

$$(SCP) \quad \begin{cases} dY(t) &= AY(t)dt + F(Y(t))dt + G(Y(t))dW_H(t), \quad t \geq 0; \\ Y(0) &= Y_0, \end{cases}$$

for initial condition  $Y_0 \in L^0((\Omega, \mathcal{F}_0); E)$ , where  $(A, D(A))$  generates a strongly continuous semigroup  $(T(t))_{t \geq 0}$  on  $E$ ,  $F : D(F) \subset E \rightarrow E$  and  $G : D(G) \subset E \rightarrow \mathcal{L}(H, E)$  are strongly measurable mappings and  $W_H$  is an  $H$ -cylindrical Wiener process on a given probability space  $(\Omega, (\mathcal{F}_t)_{t \geq 0}, \mathcal{F}, \mathbb{P})$ .

We recall three concept of solution to (SCP).

**Definition 1.1.** An  $H$ -strongly measurable adapted process  $Y$  is called a *mild solution* to (SCP) if  $Y$  for all  $t > 0$  we have

- (i) there exists the Bochner integral  $\int_0^t T(t-r)F(Y(r))dr$  a.s.(almost surely);
- (ii) there exists the stochastic integral  $\int_0^t T(t-r)G(Y(r))dW_H(r)$ ;
- (iii) for almost all  $\omega$

$$(1) \quad Y(t) = T(t)Y_0 + \int_0^t T(t-r)F(Y(r))dr + \int_0^t T(t-r)G(Y(r))dW_H(r).$$

We introduce the following concept of weak solution to (SCP) which is slightly more general than the one considered in [18] and [8].

**Definition 1.2.** An  $H$ -strongly measurable adapted process  $Y$  is called a *weak solution* to (SCP) if  $Y$  is a.s. locally Bochner integrable and for all  $t > 0$  and  $x^* \in D(A^\odot)$ :

- (i)  $\langle F(Y), x^* \rangle$  is integrable on  $[0, t]$  a.s.;
- (ii)  $G^*(Y)x^*$  is stochastically integrable on  $[0, t]$ ;

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(iii) for almost all  $\omega$

$$\langle Y(t) - Y_0, x^* \rangle = \int_0^t \langle Y(s), A^\odot x^* \rangle ds + \int_0^t \langle F(Y(s)), x^* \rangle ds + \int_0^t G^*(Y(s)) x^* dW_H(s).$$

In next interpretation of solution to (SCP) we use the theory of stochastic integration in a Banach space as given in [16].

**Definition 1.3.** A strongly measurable adapted process  $Y$  is called a *generalized strong solution* to (SCP) if  $Y$  is a.s. locally Bochner integrable and for all  $t > 0$ :

- (i)  $\int_0^t Y(s) ds \in D(A)$  a.s.,
- (ii)  $F(Y)$  is Bochner integrable in  $[0, t]$  a.s.,
- (iii)  $G(Y)$  is stochastically integrable on  $[0, t]$ ,

and

$$Y(t) - Y_0 = A \int_0^t Y(s) ds + \int_0^t F(Y(s)) ds + \int_0^t G(Y(s)) dW_H(s) \quad a.s.$$

A process  $Y$  satisfying Definition 1.3 is called a *strong solution* to (SCP) if in addition  $Y(s) \in D(A)$  a.s. for all  $t > 0$  (see [8]). The additional condition in the definition of strong solution is not appropriate for stochastic delay equations (see Remark 4.10 in [6]) which we consider in Section 5, thus we are not focus on strong solutions to (SCP) in this paper.

The equivalence of the these three interpretation of solution to (SCP) in Hilbert space has been proved by Chojnowska-Michalik in [5], see also [8, Theorem 6.5] and [12, Theorem 9.15]. For a linear (SCP) with additive noise in a Banach space the equivalence of weak and mild solution is given in [4], while in [14, Theorems 8.6 and 8.10] one can find the proof of equivalence of these three concept of solution. In [19] non-autonomous stochastic Cauchy problem in a  $UMD^-$  Banach space is analysed, Veraar consider three interpretations of solution: mild, variational and weak. Applying the stochastic Fubini theorem he prove that mild and variational interpretations are identical. Moreover, only for reflexive Banach spaces using Ito's formula it is shown in [19] that weak and variational are equal. In [6] the authors consider linear stochastic Cauchy problems ((SCP) with  $F = 0$ ) in a  $UMD^-$  Banach space and formulate sufficient conditions for equivalence of mild and generalised strong solutions of (SCP) (see Theorem 3.2 in [6]).

We complement these results showing in Sections 3 and 4 that in a class of  $UMD^-$  Banach spaces, which also contains non-reflexive  $L^1$  Banach spaces, these three concept of solutions to (SCP) are equivalent. The equivalence between mild, weak and generalised strong solutions to (SCP) with  $A$  being a generator of delay semigroup is use in [11, Theorems 3.2, 3.6] to prove the Markov representation of stochastic delay equations in  $E \times L^p(-1, 0; E)$  for some  $p \geq 1$  (see [6, Theorem 4.8]). In Section 5 we apply this result to prove existence and continuity of weak solutions to stochastic delay equations with additive noise in a Banach space.

It is worth mentioning that it turns out that for stochastic evolution equations with non-additive noise in a  $UMD$  Banach space which are not type 2 it is convenient to analyse a concept of mild  $E_\eta$ -solution of (SCP). This interpretation is more general than those considered in the article. The existence, uniqueness and Hölder regularity of mild  $E_\eta$ -solution to (SCP) with  $A$  being an analytic generator has been proved in [17]. Since the delay semigroup is not an analytic semigroup, we can not use these results in Section 5.

Mainly based on [16] in the next section we present sufficient conditions for the existence of stochastic integral in a  $UMD^-$  Banach space, and some preliminary lemmas which will be useful in the sequel.

## 2. PRELIMINARIES

In the sequel we use the notation:  $E$  is a real Banach space, let  $H$  denote a separable Hilbert space,  $W_H$  is an  $H$ -cylindrical Wiener process on a given probability space  $(\Omega, (\mathcal{F}_t)_{t \geq 0}, \mathcal{F}, \mathbb{P})$ .

Let  $A : D(A) \subset E \rightarrow E$  be a generator of a  $C_0$ -semigroup  $(T(t))_{t \geq 0}$  on a Banach space  $E$ . In the case where  $E$  is not reflexive the adjoint semigroup  $(T^*(t))_{t \geq 0}$  is not necessary strongly continuous (cf. [9]). However *sun dual semigroup*  $(T^\odot(t))_{t \geq 0}$  defined as subspace semigroup by  $T^\odot(t) = T^*(t)|_{E^\odot}$  defined on  $E^\odot = \overline{D(A^*)}$  is strongly continuous (see. 2.6 in [9] and Chapter 1 in [13]). A generator  $(A^\odot, D(A^\odot))$  of the sun dual semigroup is given by  $A^\odot = A^*|_{E^\odot}$  and  $D(A^\odot) = \{x^* \in D(A^*) : A^*x^* \in E^\odot\}$ .

**Lemma 2.1.** *For all  $n \geq 1$  the set  $D((A^\odot)^n)$  separate points in  $E$ .*

*Proof.* Set  $n \geq 1$ . By strong continuity  $D((A^\odot)^n)$  is dense in  $E^\odot$  (see Proposition 1.8 w [9]), hence it is also  $*$ -weak dense in  $E^\odot$ . From Theorem 1.3.1 in [13] it follows that  $E^\odot$  is  $*$ -weak dense in  $E^*$ . Thus  $D((A^\odot)^n)$  is  $*$ -weak dense in  $E^*$ .  $\square$

Let  $W_H$  be an  $H$ -cylindrical Brownian motion and let  $F : D(F) \subset E \rightarrow E$  and  $G : D(G) \subset E \rightarrow \mathcal{L}(H, E)$  satisfies the conditions:

(HA)  $D(F)$  is dense in  $E$  and there exists  $a \in L^1_{loc}(0, \infty)$  such that for all  $t > 0$  and  $x, y \in D(F)$  we have

$$\begin{aligned} \|T(t)F(x)\|_E &\leq a(t)(1 + \|x\|_E), \\ \|T(t)(F(x) - F(y))\|_E &\leq a(t)\|x - y\|_E, \end{aligned}$$

(HB)  $D(G)$  is dense in  $E$  and there exists  $b \in L^2_{loc}(0, \infty)$  such that for all  $t > 0$  and  $x, y \in D(G)$  we have

$$\begin{aligned} \|T(t)G(x)\|_{\gamma(H, E)} &\leq b(t)(1 + \|x\|_E), \\ \|T(t)(G(x) - G(y))\|_{\gamma(H, E)} &\leq b(t)\|x - y\|_E. \end{aligned}$$

The symbol  $\gamma(H, E)$  stand for the space of  $\gamma$ -radonifying linear operators from  $H$  to  $E$ . The space  $\gamma(H, E)$  is defined to be the closure of the finite rank operators under the norm

$$\|R\|_{\gamma(H, E)}^2 := \sup_{(h_j)_{j=1}^k} \mathbb{E} \left\| \sum_{j=1}^k \gamma_j R h_j \right\|_E^2,$$

where the supremum is taken over all finite orthonormal systems  $h = (h_j)_{j=1}^k$  in  $H$  and  $(\gamma_j)_{j \geq 1}$  is a sequence of independent standard Gaussian random variables. Hence  $\gamma(H, E)$  is a separable Banach space.

**Lemma 2.2.** *If (HA) and (HB) hold, then for all  $x^* \in D(A^*)$  there exist constants  $C_1(x^*)$ ,  $C_2(x^*) > 0$  such that*

$$\begin{aligned} |\langle F(x), x^* \rangle| &\leq C_1(x^*)(1 + \|x\|_E), \\ |\langle F(x) - F(y), x^* \rangle| &\leq C_1(x^*)\|x - y\|_E \end{aligned}$$

for all  $x, y \in D(F)$ , and

$$\begin{aligned} \|G^*(x)x^*\|_H &\leq C_2(x^*)(1 + \|x\|_E), \\ \|(G^*(x) - G^*(y))x^*\|_H &\leq C_2(x^*)\|x - y\|_E \end{aligned}$$

for all  $x, y \in D(G)$ .

*Proof.* In the case  $E$  is a Hilbert space see Lemma 9.13 in [12]. If  $E$  is a Banach space then we can repeat the reasoning from the proof of Lemma 9.13. Indeed, by strong continuity of  $(T(t))_{t \geq 0}$  and (HA) there exists  $\lambda > 0$  such that for all  $x \in D(F)$  we have the following inequality

$$\begin{aligned}
 \|(\lambda I - A)^{-1}F(x)\|_E &\leq \int_0^\infty e^{-\lambda t} \|T(t)F(x)\|_E dt \\
 (2) \quad &\leq \left( \int_0^1 a(t)dt + a(1) \int_1^\infty e^{-\lambda t} \|T(t-1)\|_{\mathcal{L}(E)} dt \right) (1 + \|x\|_E) \\
 &= C_1(1 + \|x\|_E).
 \end{aligned}$$

Since  $(A, D(A))$  is closed and densely defined,  $(\lambda I - A^*)^{-1} = ((\lambda I - A)^{-1})^*$  (cf. B.11-12 in [9]). Hence  $D(A^*) = D((\lambda I - A)^*) = (\lambda I - A^*)^{-1}(E^*)$ . Then for all  $x^* \in D(A^*)$  there exist  $y^* \in E^*$  such that  $x^* = (\lambda I - A^*)^{-1}y^*$ . By (2) we obtain

$$|\langle F(x), x^* \rangle| = |\langle (\lambda I - A)^{-1}F(x), y^* \rangle| \leq \|y^*\|_{E^*} C_1(1 + \|x\|_E).$$

Between the same lines from (HB) we get for every  $x \in D(G)$  and all  $h \in H$  the estimate

$$\begin{aligned}
 (3) \quad \|(\lambda I - A)^{-1}G(x)h\|_E &\leq \int_0^\infty e^{-\lambda t} \|T(t)G(x)\|_{\gamma(H, E)} dt \|h\|_H \\
 &\leq C_2(1 + \|x\|_E) \|h\|_H,
 \end{aligned}$$

where  $C_2 = \int_0^1 b(t)dt + b(1) \int_1^\infty e^{-\lambda t} \|T(t-1)\|_{\mathcal{L}(E)} dt$ . Hence we obtain for all  $x^* = (\lambda I - A^*)^{-1}y^* \in D(A^*)$

$$\begin{aligned}
 \|G^*(x)x^*\|_H &= \sup_{h \in H, \|h\| \leq 1} |[h, G^*(x)x^*]_H| = \sup_{h \in H, \|h\| \leq 1} |\langle G(x)h, x^* \rangle| \\
 (4) \quad &\leq \|y^*\|_{E^*} C_2(1 + \|x\|_E).
 \end{aligned}$$

□

In the sequel we use the theory of stochastic integral for  $\mathcal{L}(H, E)$ -valued process as introduced in [16]. For a Banach space with UMD property one may characterise stochastic integrability in terms of  $\gamma$ -radonifying norm. UMD property stands for Unconditional Martingale Difference property and it requires that all  $L^p(\Omega; E)$ -convergence,  $E$ -valued sequence of martingale difference are unconditionally convergent (see [10] and [16]). A  $H$ -strongly measurable adapted process  $\Psi : [0, t] \times \Omega \rightarrow \mathcal{L}(H, E)$  is stochastically integrable with respect to the cylindrical Wiener process  $W_H$  if and only if  $\Psi$  represents  $\gamma(L^2(0, t; H); E)$ -valued random variable  $R_\Psi$  given by

$$(5) \quad \langle R_\Psi f, x^* \rangle = \int_0^t \langle \Psi(s)f(s), x^* \rangle ds \quad \text{a.s.},$$

for every  $f \in L^2(0, t; H)$  and for all  $x^* \in E^*$ . For the sake of simplicity we shall say then that the process  $\Psi$  is in  $\gamma(L^2(0, t; H); E)$  a.s. (see also Lemma 2.5, 2.7 and Remark 2.8 in [16]). We have also the Burkholder-Gundy-Davies inequality:

$$(6) \quad \mathbb{E} \sup_{s \in [0, t]} \left\| \int_0^s \Psi(u) dW_H \right\|_E^p \approx_p \mathbb{E} \|R_\Psi\|_{\gamma(L^2(0, t; H), E)}^p$$

for all  $p > 0^1$ .

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<sup>1</sup>For reals  $A, B$  we use the notation  $A \lesssim_p B$  to express the fact that there exists a constant  $C > 0$ , depending on  $p$ , such that  $A \leq CB$ . We write  $A \approx_p B$  if  $A \lesssim_p B \lesssim_p A$ .

In [10] it is shown that UMD property can be characterise in terms of two properties:  $\text{UMD}^-$  and  $\text{UMD}^+$ .

**Definition 2.3.** A Banach space  $E$  has  $\text{UMD}^-$  property, if for every  $1 < p < \infty$  there exists  $\beta_p^- > 0$  such that for all  $E$ -valued sequence of  $L^p$ -martingale difference  $(d_n)_{n=1}^N$  and for all Rademacher sequence  $(r_n)_{n=1}^N$  independent from  $(d_n)_{n=1}^N$  we have the following inequality

$$(\text{UMD}^-) \quad \mathbb{E} \left\| \sum_{n=1}^N d_n \right\|_E^p \leq \beta_p^- \mathbb{E} \left\| \sum_{n=1}^N r_n d_n \right\|_E^p.$$

A Banach space  $E$  has  $\text{UMD}^+$  property, if the reverse inequality to  $(\text{UMD}^-)$  holds. Recall that class of UMD Banach spaces is in class of reflexive spaces and includes Hilbert spaces and  $L^p$  spaces for  $p \in (1, \infty)$ . Moreover, class of  $\text{UMD}^-$  Banach spaces includes also non-reflexive  $L^1$  spaces.

To integrate processes with values in  $L^1$  one needs a weaken notion of stochastic integral. In a Banach space  $E$  with  $\text{UMD}^-$  property the condition:  $\Psi$  is in  $\gamma(L^2(0, t; H), E)$  a.s. is sufficient for the process  $\Psi$  to be stochastic integrable (cf. [17]).

### 3. EQUIVALENCE OF WEAK AND MILD SOLUTIONS

**Theorem 3.1.** In a  $\text{UMD}^-$  Banach space  $E$  consider (SCP) with  $F$  and  $G$  satisfying conditions (HA) and (HB) from Section 2, respectively. Let  $Y$  be an  $E$ -valued  $H$ -strongly measurable adapted process with almost surely locally Bochner square integrable trajectories. If for all  $t > 0$  the process:

$$(7) \quad u \mapsto T(t-u)G(Y(u))$$

is in  $\gamma(L^2(0, t; H), E)$  a.s., then  $Y$  is a weak solution to (SCP) if and only if  $Y$  is a mild solution to (SCP) i.e.  $Y$  satisfies, for all  $t \geq 0$ ,

$$(8) \quad Y(t) = T(t)Y_0 + \int_0^t T(t-s)F(Y(s))ds + \int_0^t T(t-s)G(Y(s))dW_H(s) \quad \text{a.s.}$$

Before proving the theorem, we formulate some consequences of Lemma 2.2.

**Remark 3.2.** Fix  $x^* \in D(A^*)$ . Let  $Y$  be a  $E$ -valued, strongly measurable adapted process with locally Bochner square integrable trajectories a.s. Then,

- (i) condition (HA) and Lemma 2.2 implies that  $E \ni x \mapsto \langle F(x), x^* \rangle \in \mathbb{R}$  is a Lipschitz-continuous function. Hence the condition (i) from the definition of weak solution to (SCP) is satisfied.
- (ii) from (HB) and Lemma 2.2 it follows that  $E \ni x \mapsto G^*(x)x^* \in H$  is a Lipschitz-continuous function. Hence the process  $G^*(Y)x^*$  is strongly measurable and adapted with locally square integrable trajectories a.s. In particular  $G^*(Y)x^*$  is stochastically integrable on  $[0, t]$  for all  $t > 0$ .
- (iii) by (HA) and (HB) the mappings  $E \ni x \mapsto T(s)F(x) \in E$ ,  $E \ni x \mapsto T(s)G(x) \in \gamma(H, E)$  are continuous functions, hence processes

$$T(t-\cdot)F(Y(\cdot)), \quad T(t-\cdot)G(Y(\cdot))$$

are adapted, strongly and  $H$ -strongly measurable, respectively. Moreover, the first process has trajectories locally Bochner integrable a.s.

- (iv) since the process (7) represents an element from  $L^0(\Omega; \gamma(L^2(0, t; H); E))$  and  $E$  has the  $\text{UMD}^-$  property, the stochastic integral in (8) is well defined.

*Proof of Theorem 3.1.* We apply the stochastic Fubini theorem form [15] to obtain the key equations for the proof of Theorem 3.1, equations (11), (12) and (14) below. As every adapted and measurable process with values in Polish space has a progressive version we may assume that  $Y$  is progressive. Moreover, observe that as  $Y$  is strongly measurable one may assume without loss of generality that  $E$  is separable.

**Step 1.** Fix  $x^* \in D(A^*)$  and  $t > 0$ . Consider the processes:

$$\begin{aligned}\Psi_1^* x^* : [0, t]^2 \times \Omega &\rightarrow H, & \Psi_1^* x^*(s, u, \omega) &= 1_{[0, s]}(u) G^*(Y(u, \omega)) T^*(t - s) x^*, \\ \Psi_2^* x^* : [0, t]^2 \times \Omega &\rightarrow H, & \Psi_2^* x^*(s, u, \omega) &= 1_{[0, s]}(u) G^*(Y(u, \omega)) T^*(s - u) x^*,\end{aligned}$$

which are formed from  $\mathcal{L}(H, E)$ -valued processes  $\Psi_1, \Psi_2$  given by

$$\Psi_1(s, u, \omega) := 1_{[0, s]}(u) T(t - s) G(Y(u, \omega)), \Psi_2(s, u, \omega) := 1_{[0, s]}(u) T(s - u) G(Y(u, \omega)).$$

By Remark 3.2.(iii) it follows that  $\Psi_1, \Psi_2$  are  $H$ -strongly measurable. As  $Y$  is assumed to be progressive we conclude that for all  $s \in [0, t]$  and  $h \in H$  the selections:  $(\Psi_1)_s h(u, \omega) := \Psi_1(s, u, \omega)h$ ,  $(\Psi_2)_s h := \Psi_2(s, u, \omega)h$  are progressive. Hence by Proposition 2.2 in [16] we obtain strong measurability of  $\Psi_1^* x^*$ ,  $\Psi_2^* x^*$  and for all  $s \in [0, t]$  progressive measurability of  $\Psi_1^* x^*(s)$ ,  $\Psi_2^* x^*(s)$ . To apply the stochastic Fubini theorem (Theorem 3.5 in [15]) to  $\Psi_1^* x^*$ ,  $\Psi_2^* x^*$  it is enough to show that

$$\int_0^t \left( \int_0^t \|\Psi_1^* x^*\|_H^2 du \right)^2 ds, \int_0^t \left( \int_0^t \|\Psi_2^* x^*\|_H^2 du \right)^2 ds < \infty \quad \text{a.s.}$$

Indeed, using Lemma 2.2 we have the following estimate

$$\begin{aligned}(9) \quad \int_0^t \left( \int_0^t \|\Psi_2^* x^*\|_H^2 du \right)^{\frac{1}{2}} ds &= \int_0^t \left( \int_0^s \|G^*(Y(u)) T^*(s - u) x^*\|_H^2 du \right)^{\frac{1}{2}} ds \\ &\leq \int_0^t \left( \int_0^s C^2(s - u) (1 + \|Y(u)\|_E^2) du \right)^{\frac{1}{2}} ds \quad \text{a.s.},\end{aligned}$$

where  $C(s - u) = \|y_{s-u}^*\|_{E^*} C_2$  is the constant occurring in inequality (4) for  $y_{s-u}^* \in E^*$  such that  $(\lambda I - A^*)^{-1} y_{s-u}^* = T^*(s - u) x^*$ . Since  $T^*(s - u) x^* \in D(A^*)$  and  $A^* T^*(s - u) x^* = T^*(s - u) A^* x^*$  (see Proposition 1.2.1 in [13]), we have

$$(10) \quad C(s - u) = C_2 \|T^*(s - u)(\lambda I - A^*) x^*\|_{E^*} \leq C_2 M_T(t) \|(\lambda I - A^*) x^*\|_{E^*},$$

where  $M_T(t) = \sup_{s \in [0, t]} \|T(s)\|_{\mathcal{L}(E)}$ . Hence combining (10) and (9) we obtain, almost surely,

$$\int_0^t \left( \int_0^t \|\Psi_2^* x^*\|_H^2 du \right)^{\frac{1}{2}} ds \leq t C_2 M_T(t) \|(\lambda I - A^*) x^*\|_{E^*} \left( \sqrt{t} + \|Y(s)\|_{L^2(0, t; E)} \right).$$

In the similar way we get

$$\int_0^t \left( \int_0^t \|\Psi_1^* x^*\|_H^2 du \right)^{\frac{1}{2}} ds \leq t C_2 M_T(t) \|(\lambda I - A^*) x^*\|_{E^*} \left( \sqrt{t} + \|Y(s)\|_{L^2(0, t; E)} \right).$$

Thus from stochastic Fubini's theorem it follows that

$$(11) \quad \int_0^t \int_0^s G^*(Y(u)) T^*(t - s) x^* dW_H(u) ds = \int_0^t \int_0^t \Psi_1^* x^* ds dW_H(u),$$

$$(12) \quad \int_0^t \int_0^s G^*(Y(u)) T^*(s - u) x^* dW_H(u) ds = \int_0^t \int_0^t \Psi_2^* x^* ds dW_H(u).$$

Moreover, notice that for all  $u \in [0, t]$

$$(13) \quad \text{we obtain } \int_0^t \Psi_1^* x^* ds = \int_0^t \Psi_2^* x^* ds = \int_0^{t-u} G^*(Y(u, \omega)) T^*(s) x^* ds \quad \text{a.s.}$$

By (13) and strong continuity of  $(T^\odot(t))_{t \geq 0}$  it follows that for all  $x^* \in D(A^\odot)$  we have, almost surely,

$$(14) \quad \begin{aligned} \int_0^t \int_0^t \Psi_1^* A^\odot x^* ds dW_H(u) &= \int_0^t \int_0^t \Psi_2^* A^\odot x^* ds dW_H(u) \\ &= \int_0^t \int_0^{t-u} G^*(Y(u)) T^\odot(s) A^\odot x^* ds dW_H(u) \\ &= \int_0^t G^*(Y(u)) \int_0^{t-u} T^\odot(s) A^\odot x^* ds dW_H(u) \\ &= \int_0^t G^*(Y(u)) (T^\odot(t-u)x^* - x^*) dW_H(u) \end{aligned}$$

Notice that by Remark 3.2.(iv) the stochastic integral on the right hand side of (14) is well defined.

**Step 2.** Let us suppose that  $Y$  is a weak solution to (SCP), we prove that (8) holds. By (14) i (11) and by the definition of a weak solution for all  $x^* \in D(A^\odot)$  and  $t > 0$  one has, almost surely,

$$(15) \quad \begin{aligned} \langle Y(t) - Y_0, x^* \rangle - \int_0^t \langle Y(s), A^\odot x^* \rangle ds - \int_0^t \langle F(Y(s)), x^* \rangle ds &= \int_0^t G^*(Y(u)) x^* dW(u) \\ &\stackrel{(14)}{=} \int_0^t G^*(Y(u)) T^\odot(t-u) x^* dW_H(u) - \int_0^t \int_0^t \Psi_1^* A^\odot x^* ds dW_H(u) \\ &\stackrel{(11)}{=} \int_0^t G^*(Y(u)) T^\odot(t-u) x^* dW_H(u) \\ &\quad - \int_0^t \int_0^s G^*(Y(u)) T^\odot(t-s) A^\odot x^* dW_H(u) ds. \end{aligned}$$

Assuming that  $y^* = A^\odot x^* \in D(A^\odot)$  and using the definition of a weak solution again, we can write the last term in (15) as follows

$$(16) \quad \begin{aligned} \int_0^t \int_0^s G^*(Y(u)) T^\odot(t-s) A^\odot x^* dW_H(u) ds &= \int_0^t \left[ \langle Y(s) - Y_0, T^\odot(t-s) y^* \rangle \right. \\ &\quad \left. - \int_0^s \langle F(Y(u)), T^\odot(t-s) y^* \rangle du - \int_0^s \langle Y(u), A^\odot T^\odot(t-s) y^* \rangle du \right] ds \\ &= - \int_0^t \langle Y_0, T^\odot(t-s) y^* \rangle ds + \int_0^t \langle Y(s), y^* \rangle ds \\ &\quad - \int_0^t \int_u^t \langle F(Y(u)), T^\odot(t-s) y^* \rangle ds du \quad \text{a.s.}, \end{aligned}$$

where the first equality follows from condition (iii) of the definition of a weak solution to (SCP) and in the last equality we use strong continuity of  $(T^\odot(t))_{t \geq 0}$  and Fubini's theorem. Applying (16) into

(15) for all  $x^* \in D((A^\odot)^2)$  one has, almost surely,

$$\begin{aligned}
\langle Y(t), x^* \rangle - \langle Y_0, x^* \rangle &= \left\langle \int_0^t Y(s) ds, A^\odot x^* \right\rangle - \int_0^t \langle F(Y(s)), x^* \rangle ds \\
&= \int_0^t G^*(Y(u)) T^\odot(t-u) x^* dW_H(u) + \int_0^t \langle Y_0, T^\odot(t-s) A^\odot x^* \rangle ds \\
&\quad - \int_0^t \langle Y(s), A^\odot x^* \rangle ds + \int_0^t \int_u^t \langle F(Y(u)), T^\odot(t-s) A^\odot x^* \rangle ds du \\
&= \int_0^t G^*(Y(u)) T^\odot(t-u) x^* dW_H(u) + \langle Y_0, T^\odot(t) x^* \rangle - \langle Y_0, x^* \rangle \\
&\quad - \int_0^t \langle Y(s), A^\odot x^* \rangle ds + \int_0^t \langle F(Y(u)), T^\odot(t-u) x^* \rangle du - \int_0^t \langle F(Y(u)), x^* \rangle du,
\end{aligned}$$

By Remark 3.2.(iii)-(iv) the process  $T(t-\cdot)G(Y(\cdot))$  is stochastically integrable on  $[0, t]$  and  $T(t-\cdot)F(Y(\cdot))$  has, almost surely, trajectories Bochner integrable on  $[0, t]$ . Hence for all  $x^* \in D((A^\odot)^2)$  one has, almost surely,

$$\begin{aligned}
\langle Y(t), x^* \rangle &= \langle T(t)Y_0, x^* \rangle \\
(17) \quad &+ \left\langle \int_0^t T(t-u)F(Y(u))du, x^* \right\rangle + \left\langle \int_0^t T(t-u)G(Y(u))dW_H(u), x^* \right\rangle.
\end{aligned}$$

Notice that by Lemma 2.1 the set  $D((A^\odot)^2)$  separates the points of  $E$  and  $E$  is assume to be separable, hence by the Hahn-Banach theorem there exists a sequence  $(x_n^*)_{n \geq 1}$  of elements from  $D((A^\odot)^2)$  which separates the points of  $E$ . Thus (17) holds simultaneously for all  $x_n^*$  on set of measure one. Therefore (8) holds.

On the other hand assume that  $Y$  is a mild solution to (SCP). By Remark 3.2.(ii) it follows that for all  $x^* \in D(A^*)$  the integral  $\int_0^t G^*(Y(u))x^*dW_H(u)$  is well defined. Moreover, by (8) and then by Fubini's theorem and (14), and once more by (8) we obtain, almost surely,

$$\begin{aligned}
\left\langle \int_0^t Y(s) ds, A^\odot x^* \right\rangle &\stackrel{(8)}{=} \left\langle \int_0^t T(s)Y_0 ds, A^\odot x^* \right\rangle + \left\langle \int_0^t \int_0^s T(s-u)F(Y(u))duds, A^\odot x^* \right\rangle \\
&\quad + \int_0^t \int_0^s G^*(Y(u))T^\odot(s-u)A^\odot x^* dW_H(u) ds \\
&\stackrel{(14)}{=} \langle T(s)Y_0, x^* \rangle - \langle Y_0, x^* \rangle + \left\langle \int_0^t (T(t-u)F(Y(u))du, x^* \right\rangle - \int_0^t \langle F(Y(u)), x^* \rangle du \\
&\quad + \left\langle \int_0^t T(t-u)G(Y(u))dW_H(u), x^* \right\rangle - \int_0^t G^*(Y(u))x^* dW_H(u) \\
&\stackrel{(8)}{=} \langle Y(t), x^* \rangle - \langle Y(0), x^* \rangle - \int_0^t \langle F(Y(u)), x^* \rangle du - \int_0^t G^*(Y(u))x^* dW_H(u).
\end{aligned}$$

□

#### 4. EQUIVALENCE OF GENERALISED STRONG, WEAK AND MILD SOLUTIONS

In [6] a generalised strong solution to (SCP) is defined and its equivalence to a mild solution of SCP is proven. Under weaker assumptions we establish in Theorem 4.1 equivalence of mild, weak and generalised strong solutions. Let us extend the condition (HA)



(HA') Assume that (HA) is satisfied and for all  $t > 0$  and  $g \in L^1(0, t; E)$  the function  $F(g)$  is Bochner integrable on  $[0, t]$ .

It is clear that if  $F$  is a Lipschitz function, then (HA') is satisfied.

**Theorem 4.1.** *Assume that  $E$  has  $\text{UMD}^-$  property and hypotheses (HA') and (HB) are satisfied. Let  $Y$  be an  $E$ -valued  $H$ -strongly measurable adapted process with locally Bochner square integrable trajectories a.s. If for all  $t > 0$  the processes:*

$$(18) \quad u \mapsto G(Y(u)), \quad u \mapsto T(t-u)G(Y(u)), \quad u \mapsto \int_0^{t-u} T(s)G(Y(u, \omega))ds$$

are in  $\gamma(L^2(0, t; H), E)$  a.s., then the following condition are equivalent:

- (i)  $Y$  is a generalised strong solution of (SCP).
- (ii)  $Y$  is a weak solution of (SCP).
- (iii)  $Y$  is a mild solution of (SCP).

In the case where  $E$  is reflexive the proof of the Theorem 4.1 is simple consequence of Theorem 3.1 (see Remark 4.2 below).

**Remark 4.2.** (i) *It is obvious that if  $Y$  is a generalised strong solution to (SCP), then  $Y$  is a weak solution of (SCP).*

(ii) *Let  $E$  has  $\text{UMD}$  property and  $(T(t))_{t \geq 0}$  is  $\gamma$ -bounded. If  $G(Y(\cdot))$  is in  $\gamma(L^2(0, t; H), E)$  a.s., then the processes  $T(t-\cdot)G(Y(\cdot))$ ,  $\int_0^{t-\cdot} T(s)G(Y(\cdot, \omega))ds$  are in  $\gamma(L^2(0, t; H), E)$  a.s. For the definition of  $\gamma$ -boundness cf. [17]. An analytic semigroup is an example of  $\gamma$ -bounded semigroup.*

(iii) *Let  $E$  be a reflexive Banach space, and for all  $t > 0$  the process  $G(Y(\cdot))$  is in  $\gamma(L^2(0, t; H), E)$  a.s. Assume that (HA'), (HB) are satisfied and  $Y$  has almost all trajectories locally square integrable. Then, it is easy to prove that a weak solution to (SCP) is a generalised strong solution to (SCP). Indeed, let  $Y$  be a weak solution to (SCP), then for every  $t > 0$  and all  $x^* \in D(A^*)$  we have the equality*

$$(19) \quad \left\langle \int_0^t Y(s)ds, A^* x^* \right\rangle = \langle e(t), x^* \rangle \quad \text{a.s.},$$

where  $e(t) = Y(t) - Y_0 - \int_0^t F(Y(s))ds - \int_0^t G(Y(s))ds \in E \subset E^{**}$ . From reflexivity of  $E$  it follows that  $D(A^*)$  is dense in  $E^*$ , hence, almost surely, the right hand side of (19) has an extension to bounded linear functional  $\phi$  on  $E^*$ . Thus and by the definition of  $A^*$  one has  $\int_0^t Y(s)ds \in D(A^{**})$  and  $A^{**} \int_0^t Y(s)ds = e(t)$  a.s. Finally, by reflexivity of  $E$  we can replace in the last equality  $(A^{**}, D(A^{**}))$  by  $(A, D(A))$  and the assertion follows (see B.10 w [9]).

*Proof of Theorem 4.1.* By Theorem 3.1 and Remark 4.2.(i) it is enough to prove that every mild solution to (SCP) is a generalised strong solution of (SCP).

Fix  $t > 0$ . Let  $Y$  be a mild solution of (SCP) satisfying the assumptions of Theorem 4.1. Observe that  $[0, t] \times \Omega \ni (u, \omega) \mapsto \int_0^t \Psi_1(s, u, \omega)ds$ , where  $\Psi_1$  is a process defined in Step 1 of the proof of Theorem 3.1, satisfies the assumptions of Lemma 2.8 in [6] i.e. for all  $h \in H$  process  $\Phi_1(s)h \in D(A)$

a.s. and the processes

$$\begin{aligned} u &\mapsto \int_0^t \Psi_1(s, u, \omega) ds = \int_0^{t-u} T(s)G(Y(u))ds \in D(A), \\ u &\mapsto A \int_0^t \Psi_1(s, u, \omega) ds = T(t-u)G(Y(u)) - G(Y(u)) \text{ a.s.} \end{aligned}$$

represents elements in  $\gamma(L^2(0, t; H), E)$  a.s. Hence from Lemma 2.8 in [6] it follows that

$$\begin{aligned} (20) \quad & \int_0^t \int_0^{t-u} T(s)G(Y(u))ds dW_H(u) \in D(A), \\ & A \int_0^t \int_0^{t-u} T(s)G(Y(u))ds dW_H(u) = \int_0^t [T(t-u)G(Y(u)) - G(Y(u))]dW_H(u) \text{ a.s.} \end{aligned}$$

Moreover, by (HA') and the properties of strongly continuous semigroup we obtain, almost surely,

$$\begin{aligned} & \int_0^t \int_0^{t-u} T(s)F(Y(u))ds du, \int_0^t T(s)Y_0 ds \in D(A), \\ & A \int_0^t \int_0^{t-u} T(s)F(Y(u))ds du = \int_0^t T(t-u)F(Y(u))du - \int_0^t F(Y(u))du, \\ (21) \quad & A \int_0^t T(s)Y_0 ds = T(t)Y_0 - Y_0. \end{aligned}$$

Therefore, by (8) and (20)-(21) we have

$$\begin{aligned} & \int_0^t Y(s)ds \in D(A), \\ & A \int_0^t Y(s)ds \stackrel{(8), (20)-(21)}{=} T(t)Y_0 - Y_0 + \int_0^t T(t-u)F(Y(u))du - \int_0^t F(Y(u))du \\ & \quad + \int_0^t [T(t-u)G(Y(u)) - G(Y(u))]dW_H(u) \\ & \stackrel{(8)}{=} Y(t) - Y_0 - \int_0^t F(Y(u))du - \int_0^t G(Y(u))du \text{ a.s.} \end{aligned}$$

□

## 5. THE STOCHASTIC EVOLUTION EQUATIONS WITH ADDITIVE NOISE

In a separable Banach space  $E$  consider the following version of (SCP), where the noise is introduced additively i.e.  $G \in \mathcal{L}(H, E)$  and

$$(SCP_a) \quad \begin{cases} dY(t) &= AY(t)dt + F(Y(t))dt + GdW_H(t), \quad t \geq 0; \\ Y(0) &= y, \end{cases}$$

Here we do not need the assumption that  $E$  has  $UMD^-$  property, since stochastic Wiener integral in every Banach space is characterised by  $\gamma$ -norms (see [16]). From Theorem 3.1 and Theorem 4.1 we obtain the following corollaries.

**Corollary 5.1.** *Assume that condition (HA) is satisfied and  $Y$  is an  $E$ -valued  $H$ -strongly measurable adapted process with locally Bochner square integrable trajectories a.s. If for some  $t_0 > 0$  the mapping*

$$(22) \quad u \mapsto T(t_0 - u)G$$

*represents an element in  $\gamma(L^2(0, t_0; H), E)$  a.s., Then  $Y$  is a weak solution to (SCP) if and only if  $Y$  is a mild solution to (SCP) i.e.  $Y$  satisfies, for all  $t \geq 0$ ,*

$$(23) \quad Y(t) = T(t)y + \int_0^t T(t-s)F(Y(s))ds + \int_0^t T(t-s)GdW_H(s) \quad \text{a.s.}$$

**Corollary 5.2.** *Assume that condition (HA') is satisfied and  $Y$  is an  $E$ -valued  $H$ -strongly measurable adapted process and possesses locally Bochner square integrable trajectories a.s. If for some  $t > 0$  the mappings:*

$$(24) \quad u \mapsto T(t_0 - u)G, \quad u \mapsto \int_0^{t_0-u} T(s)Gds$$

*represent elements in  $\gamma(L^2(0, t; H), E)$  a.s., then the notions of generalised, weak and mild solutions to (SCP) are equivalent.*

**Remark 5.3.** *Notice that from Theorem 7.1 in [18] it follows that if there exists  $t_0 > 0$  such that  $u \mapsto T(t_0 - u)G$  and  $u \mapsto \int_0^{t_0-u} T(s)Gds$  represent elements from  $\gamma(L^2(0, t_0; H), E)$  a.s., then  $[0, t] \ni u \mapsto T(t - u)G, u \mapsto \int_0^{t-u} T(s)Gds$  also belong to  $\gamma(L^2(0, t; H), E)$  a.s. for all  $t > 0$ .*

Corollary 5.1 yields the following result concerning the existence and uniqueness of weak solution to (SCP) (see [8], [12] and [4],[18] for the linear case.)

**Theorem 5.4.** *Let  $q \geq 1$ . Assuming the hypothesis of Corollary 5.1 for every  $t > 0$  and all  $y \in L^q((\Omega, \mathcal{F}_0); E)$  we have:*

- (i) *in  $\mathbb{SL}_{\mathcal{F}}^q(0, t; E)$  there exists a unique weak solution  $Y(\cdot; y)$  to (SCP);*
- (ii) *there exists  $L > 0$  such that  $x, y \in L^q(\Omega; E)$*

$$\begin{aligned} \sup_{s \in [0, t]} \mathbb{E} \|Y(s; x)\|^q &\leq L(1 + \mathbb{E} \|x\|^q), \\ \sup_{s \in [0, t]} \mathbb{E} \|Y(s; x) - Y(s; y)\|^q &\leq L \mathbb{E} \|x - y\|^q; \end{aligned}$$

- (iii) *for all  $x \in E$  and  $s \geq 0$  the probability distribution of  $Y(s; x)$  does not depend on cylindrical Wiener process  $W_H$  and the underlying probability space.*

*Proof.* Let  $q \geq 1$  and  $t > 0$ . For  $y \in L^q((\Omega, \mathcal{F}_0); E)$  we define a mapping  $\mathcal{K}$  by

$$\mathcal{K}(Z)(s) = T(s)y + \int_0^s T(s-u)F(Z(u))du + \int_0^s T(s-u)GdW_H(u)$$

for all  $Z \in \mathbb{SL}_{\mathcal{F}}^q(0, t; E)$ . The symbol  $\mathbb{SL}_{\mathcal{F}}^q(0, t; E)$  stands for the Banach space of strongly measurable, adapted process  $Y$  with the Bielecki's type norm  $\|Y\|_\beta = \sup_{s \in [0, t]} \left( e^{-\beta s} \mathbb{E} \|Y(s)\|_{\mathcal{E}_p}^q \right)^{\frac{1}{q}}$  for some  $\beta > 0$ . By the assumptions of theorem it follows that both stochastic and Bochner integrals in the definition of  $\mathcal{K}$  are well defined. The first term of  $\mathcal{K}$  is continuous a.s. Corollary 6.5 in [18] yields that the stochastic

convolution in  $\mathcal{K}$  is continuous process in  $q$ -th moment. Moreover, by (HA) and the Minkowski's integral inequality for all  $Z \in \mathbb{SL}_{\mathcal{F}}^q(0, t; E)$  and for every  $s \in [0, t]$  one gets

$$\begin{aligned} e^{-s\beta} \left( \mathbb{E} \left\| \int_0^s T(s-u)F(Z(u))du \right\|_E^q \right)^{\frac{1}{q}} &\leq \\ &\leq e^{-s\beta} \int_0^s a(s-u)e^{\beta u}e^{-\beta u} (\mathbb{E} (1 + \|Z(u)\|_E^q)^{\frac{1}{q}}) du \\ &\leq (1 + \|Z\|_\beta) \int_0^s a(u)e^{-\beta u} du. \end{aligned}$$

Hence

$$(25) \quad \sup_{s \in [0, t]} e^{-s\beta} \left( \mathbb{E} \left\| \int_0^s T(s-u)F(Z(u))du \right\|_E^q \right)^{\frac{1}{q}} \leq C_{\beta, a} (1 + \|Z\|_\beta),$$

where  $C_{\beta, a} = \int_0^t \tilde{a}(u)e^{-\beta u} du$ . Between the same lines using (HA) for all  $Z_1, Z_2 \in \mathbb{SL}_{\mathcal{F}}^q(0, t; E)$  one has

$$(26) \quad \sup_{s \in [0, t]} e^{-s\beta} \left( \mathbb{E} \left\| \int_0^s T(s-u) (F(Z_1(u)) - F(Z_2(u))) du \right\|_E^q \right)^{\frac{1}{q}} \leq C_{\beta, a} \|Z_1 - Z_2\|_\beta.$$

Hence for  $\beta > 0$  large enough the operator  $\mathcal{K}$  is a strict contraction in  $\mathbb{SL}_{\mathcal{F}}^q(0, t; E)$ . Therefore, the existence and uniqueness results follows by the Banach fixed-point theorem and by Corollary 5.1. For the proof of part (ii) and (iii) see the proof of Theorem 9.29 in [12].  $\square$

Using the factorization method as introduced in Section 2 of [7] and Theorem 3.4 in [6] (see also Theorem 3.3 in [20]) we obtain sufficient condition for continuity of solution to (SCPa).

**Theorem 5.5.** *Let  $q > 2$ . Under the hypotheses of Theorem 5.4, if moreover there exists  $\alpha \in (\frac{1}{q}, \frac{1}{2})$  such that for all  $t > 0$  and for the function  $a$  appearing in assumption (HA) we have*

$$\begin{aligned} \int_0^t a(s)s^{-\alpha} ds &< \infty, \\ \sup_{s \in [0, t]} \|u \mapsto (s-u)^{-\alpha} T(s-u)G\|_{\gamma(L^2(0, s; H), E)} &< \infty, \end{aligned}$$

then for all  $y \in L_{\mathcal{F}_0}^q(\Omega; E)$  the weak solution  $Y = Y(\cdot; y)$  of (SCPa) belongs to  $L^q(\Omega; C([0, t]; E))$ . Moreover, there exists  $L > 0$  such that  $x, y \in L^q(\Omega; E)$

$$(27) \quad \mathbb{E} \sup_{s \in [0, t]} \|Y(s; x)\|_E^q \leq L(1 + \mathbb{E} \|x\|_{L^q(\Omega; E)}^q),$$

$$(28) \quad \mathbb{E} \sup_{s \in [0, t]} \|Y(s; x) - Y(s; y)\|_E^q \leq L \mathbb{E} \|x - y\|_{L^q(\Omega; E)}^q.$$

**5.1. Stochastic delay equations.** In a real separable Banach space  $E$  for some  $p \geq 1$  consider stochastic delay equations of the form:

$$(29) \quad \begin{cases} dX(t) = BX(t)dt + \phi(X(t), X_t)dt + \psi dW_H(t), & t > 0, \\ X(0) = x, X_0 = f, \end{cases}$$

where  $(B, D(B))$  generates a semigroup of linear operators  $(S(t))_{t \geq 0}$  on  $E$ ,  $X_t : \Omega \times [-1, 0] \rightarrow E$  is a segment process defined as  $X_t(\theta) = X(t + \theta)$   $\theta \in [-1, 0]$ ,  $\phi$  satisfies

(H $\phi$ )  $\phi : D(\phi) \subset \mathcal{E}_p \rightarrow E$ , where  $\mathcal{E}_p = E \times L^p(-1, 0; E)$ , is densely defined mapping and there exists  $a \in L^p_{loc}(0, \infty)$  such that for all  $t > 0$  and  $\mathcal{X}, \mathcal{Y} \in D(\phi)$ ,

$$\begin{aligned} \|S(t)\phi(\mathcal{X})\|_E &\leq a(t)(1 + \|\mathcal{X}\|_{\mathcal{E}_p}), \\ \|S(t)(\phi(\mathcal{X}) - \phi(\mathcal{Y}))\|_E &\leq a(t)\|\mathcal{X} - \mathcal{Y}\|_{\mathcal{E}_p}, \end{aligned}$$

and  $\psi \in \mathcal{L}(H, E)$  is such that

(H $\psi$ ) the mapping  $u \mapsto S(u)\psi$  represents an element form  $\gamma(L^2(0, t; H), E)$  a.s. for some  $t > 0$ .

In [11, Theorem 3.9] using the equivalence of solutions from Theorem 3.1 it is shown that the mapping  $X \mapsto Y = \begin{bmatrix} X \\ X_t \end{bmatrix}$  is a bijection between weak solutions  $X$  to (29) (see Definition 3.8 in [11]) and weak solutions  $Y$  to the following stochastic evolution equation in  $\mathcal{E}_p = E \times L^p(-1, 0; E)$ :

$$(30) \quad \begin{cases} dY(t) = (AY + F(Y))dt + GdW_H(t), & t > 0, \\ Y(0) = [x, f]', \end{cases}$$

where  $[x, f]'$  is a transposition of  $[x, f]$ ,  $G = [\psi, 0]' \in \mathcal{L}(H, \mathcal{E}_p)$ ,  $F : \mathcal{E}_p \rightarrow \mathcal{E}_p$ ,  $F = [\phi, 0]'$ , and  $\left(A = \begin{bmatrix} B & 0 \\ 0 & \frac{d}{d\theta} \end{bmatrix}, D(A)\right)$  is the generator of the delay semigroup  $T(t) = \begin{bmatrix} S(t) & 0 \\ S_t & T_l(t) \end{bmatrix}$ ,  $(T_l(t))_{t \geq 0}$  is the left translation semigroup on  $L^p(-1, 0; E)$  and  $S_s \in \mathcal{L}(E, L^p(-1, 0; E))$  is given by

$$(S_s x)(\theta) = \begin{cases} 0 & \theta \in (-1, -s \vee -1) \\ S(\theta + s)x & \theta \in (-s \vee -1, 0) \end{cases}$$

for all  $s \geq 0$  and all  $x \in E$  (cf. Theorem 3.25 in [2]). Hence and using Theorems 5.4 and 5.5 (see also Corollaries 3.12 and 3.13 in [11]) we obtain the proposition.

**Proposition 5.6.** *Consider (29) with  $p \geq 1$  and assumptions (H $\phi$ ), (H $\psi$ ). If  $x \in L^{p \vee q}_{\mathcal{F}_0}(\Omega; E)$  and  $f \in L^{p \vee q}_{\mathcal{F}_0}(\Omega; L^p(-1, 0; E))$  for some  $q \geq 1$ , then for all  $t > 0$*

- (i) *in  $\mathbb{S}\mathbb{L}^{p \vee q}_{\mathcal{F}}(0, t; E)$  there exists a unique weak solution  $X = X(\cdot; x, f)$  to (30) and it satisfies, almost surely,*

$$(31) \quad X(t) = S(t)x + \int_0^t S(t-s)\phi(X(s), X_s)ds + \int_0^t S(t-s)\psi dW_H(s).$$

*Moreover, the dependence of  $X(\cdot; x, f)$  on initial conditions as in Theorem 5.4.(ii) hold.*

- (ii) *for all  $x \in E$ ,  $f \in L^p(-1, 0; E)$  and  $s \geq 0$  the probability distribution of  $X(s; x, f)$  does not depend on cylindrical Wiener process  $W_H$  and the underlying probability space.*
- (iii) *if  $(q \vee p) > 2$  and there exists  $\alpha \in (\frac{1}{q}, \frac{1}{2})$  such that*

$$(32) \quad \int_0^t a(s)s^{-\alpha}ds < \infty,$$

*where the function  $a$  is defined in assumption (H $\phi$ ), and*

$$(33) \quad \sup_{s \in [0, t]} \|u \mapsto (s - u)^{-\alpha} S(s - u)\psi\|_{\gamma(L^2(0, s; H), E)} < \infty;$$

*then the weak solution  $X(\cdot; x, f)$  to (29) belongs to  $L^{p \vee q}(\Omega; C([0, t]; E))$  and the inequities of type (27)-(28) hold.*

*Proof.* In the proof we first use the equivalence of weak solutions to (30) and (29) from Theorem 3.9 in [11] and then we apply Theorems 5.4 and 5.5 to problem (30). Notice that from (H $\phi$ ) it follows that condition (HA) is satisfied for  $F = [\phi, 0]'$ . Moreover, we prove that  $[0, t] \ni u \mapsto T(u)[\psi, 0]'$

represents an operator  $[R_{\pi_1}, R_{\pi_2}]'$  in  $\gamma(L^2(0, t; H), \mathcal{E}_p)$  a.s if and only if condition (H $\psi$ ) hold. Indeed, by the properties of delay semigroup  $(T(t))_{t \geq 0}$  (see (9) in [11] and Proposition 3.11 in [2]) we have  $\pi_1 T(u)[\psi, 0]' = S(u)\psi$  and  $(\pi_2 T(u)[\psi, 0]')(\theta) = 1_{(u+\theta > 0)} S(u+\theta)\psi$  for every  $u > 0$  and a.e.(almost everywhere)  $\theta \in [-1, 0]$ . Hence by (H $\psi$ ) it follows that  $R_{\pi_1} \in \gamma(L^2(0, t; H), E)$ . Furthermore, for all  $f \in L^2(0, t; H)$  using Lemma 3.4 in [11] we obtain

$$(34) \quad \begin{aligned} (R_{\pi_2} f)(\theta) &= \left( \int_0^t \pi_2 \mathcal{T}(u)[\psi f(u), 0]' du \right)(\theta) = \int_{-\theta}^t S(u+\theta) \psi f(u) du \\ &= \int_0^t S(u) \psi P_\theta f(u) du = R_{\pi_1} P_\theta f, \quad \text{a.e. } \theta \in [-1, 0], \end{aligned}$$

where  $P_\theta \in \mathcal{L}(L^2(0, t; H))$  for all  $\theta \in [-1, 0]$  is defined by

$$(P_\theta f)(u) = 1_{(0, t+\theta)}(u) f(u-\theta) \quad \text{a.e. } u \in [0, t].$$

By  $\gamma$ -Fubini isomorphism (see Proposition 2.6 in [16])  $R_{\pi_2} \in \gamma(L^2(0, t; H), L^p(-1, 0; E))$  if and only if

$$\int_{-1}^0 \|(R_{\pi_2} \cdot)(\theta)\|_{\gamma(L^2(0, t; H), E)}^p d\theta < \infty.$$

Since  $\|P_\theta\|_{\mathcal{L}(L^2(0, t; H))} \leq 1$  and using the ideal property of  $\gamma$ -radonifying operators, from (34) we get

$$\begin{aligned} \int_{-1}^0 \|(R_{\pi_2} \cdot)(\theta)\|_{\gamma(L^2(0, t; H), E)}^p d\theta &\leq \int_{-1}^0 \|R_{\pi_1}\|_{\gamma(L^2(0, t; H), E)}^p \|P_\theta\|_{\mathcal{L}(L^2(0, t; H))}^p d\theta \\ &\leq \|R_{\pi_1}\|_{\gamma(L^2(0, t; H), E)}^p < \infty. \end{aligned}$$

To prove (iii) we show that for all  $\alpha > 0$  we have the equivalence

$$(35) \quad \sup_{s \in [0, t]} \|u \mapsto (s-u)^{-\alpha} S(s-u)\psi\|_{\gamma(L^2(0, s; H), E)} < \infty$$

$$\Updownarrow$$

$$(36) \quad \sup_{s \in [0, t]} \|u \mapsto (s-u)^{-\alpha} T(s-u)G\|_{\gamma(L^2(0, s; H), \mathcal{E}_p)} < \infty.$$

It is clear that the implication (36)  $\Rightarrow$  (35) holds. For fixed  $s \leq t$  let us denote by  $[R_{s, \alpha, \pi_1}, R_{s, \alpha, \pi_2}]'$  the operator from  $\gamma(L^2(0, s; H), \mathcal{E}_p)$  represented by

$$[0, s] \ni u \mapsto (s-u)^{-\alpha} T(s-u)G = [(s-u)^{-\alpha} S(s-u)\psi, (s-u)^{-\alpha} S_{s-u}\psi]'$$

Then, in much the same way as in (34) for all  $f \in L^2(0, s; H)$  and a.e.  $\theta \in [-1, 0]$  we get

$$\begin{aligned} (R_{s, \alpha, \pi_2} f)(\theta) &= \left( \int_0^s (s-u)^{-\alpha} \pi_2 T(s-u)G f(u) du \right)(\theta) \\ &= \int_0^{s+\theta} (s-u)^{-\alpha} S(s-u+\theta) \psi f(u) du \end{aligned}$$

Since  $(s-u)^{-\alpha} \leq (s-u+\theta)^{-\alpha}$  for all  $u \leq s+\theta$ , by the ideal property of  $\gamma$ -radonifying operators for a.e  $\theta \in [-1, 0]$  we obtain

$$\|(R_{s, \alpha, \pi_2} \cdot)(\theta)\|_{\gamma(L^2(0, s; H), E)} \leq \|(R_{s+\theta, \alpha, \pi_1} \cdot)(\theta)\|_{\gamma(L^2(0, s+\theta; H), E)}.$$

Hence

$$\begin{aligned} \int_{-1}^0 \|(R_{s,\alpha,\pi_2}(\cdot))(\theta)\|_{\gamma(L^2(0,s;H),E)}^p d\theta &\leq \int_{-1}^0 \|R_{s+\theta,\alpha,\pi_1}\|_{\gamma(L^2(0,s+\theta);H,E)}^p d\theta \\ &\leq \sup_{s \in [0,t]} \|R_{s,\alpha,\pi_1}\|_{\gamma(L^2(0,s;H),E)}^p < \infty. \end{aligned}$$

□

## 5.2. Examples.

5.2.1. *Stochastic transport equation with delay.* Let  $E = C_0([0,1]) = \{f \in C([0,1]) : f(0) = 0\}$ . Consider the following stochastic transport equation with delay in  $C_0([0,1])$ :

$$(37) \quad \begin{cases} dy(t, \xi) = \left( -\frac{\partial y(t, \xi)}{\partial \xi} - \mu y(t, \xi) \right) dt + \left[ \int_{t-1}^t \varphi(s-t, \xi) y(s, \xi) ds + f_1(y(t, \xi)) \right. \\ \quad \left. + \int_{t-1}^t k(s-t, \xi) f_2(y(s, \xi)) ds \right] dt + \psi(\xi) dW(t), \quad t \geq 0; \\ \frac{\partial y(t, 0)}{\partial \xi} = 0, \quad y(t, 0) = 0; \\ y(0, \xi) = x_0(\xi), \quad x_0 \in C_0([0,1]) \quad y(\theta, \xi) = f_0(\theta, \xi), \quad f_0 \in L^p(-1, 0; C_0([0,1])), \end{cases}$$

where  $p \geq 1$ ,  $\varphi, k \in C([0,1]; L^{p'}(-1,0))$  for some  $p' \in (1, \infty]$  such that  $\frac{1}{p} + \frac{1}{p'} = 1$ , and  $f_1, f_2 : \mathbb{R} \rightarrow \mathbb{R}$  are Lipschitz functions, and  $\psi \in C_0([0,1])$ , and  $W$  is one-dimensional Brownian motion. This equation can be used to model product goodwill (see [1] for deterministic goodwill model). Let  $B$  be a differential operator on  $E = C_0(0,1)$  such that

$$Bx = -\frac{dx}{d\xi} - \mu x, \quad D(B) = \{x \in C^1([0,1]) : x(0) = x'(0) = 0\}.$$

By [9] (see p. 86 and section 5.11) it follows that  $(B, D(B))$  generates strongly continuous nilpotent semigroup  $(S(t))_{t \geq 0}$  on  $C_0([0,1])$  such that

$$(38) \quad (S(t)x)(\xi) = \begin{cases} e^{-\xi\mu} x(\xi-t) & \xi-t \geq 0 \\ 0 & \xi-t < 0 \end{cases},$$

for all  $x \in C_0([0,1])$  and all  $s \in [0,1]$ . Let us introduce the notation:

$$(39) \quad \phi(x, h)(\xi) = \int_{-1}^0 \varphi(\theta, \xi) h(\theta, \xi) d\theta + f_1(x(\xi)) + \int_{-1}^0 k(\theta, \xi) f_2(h(\theta, \xi)) d\theta,$$

$$(40) \quad (\psi u)(\xi) = \psi(\xi) u$$

for all  $[x, h]' \in \mathcal{E}_p = C_0([0,1]) \times L^p(-1,0; C_0([0,1]))$ , and all  $u \in \mathbb{R}$ . Then, we can rewrite (37) in the form (29). Observe that  $\phi$  is Lipschitz-continuous with the Lipschitz constant

$$L = 2^{\frac{1}{p}} \left( L_{f_1} \vee (L_{f_2} \|k\|_{C([0,1]; L^{p'}(-1,0; E))} + \|\varphi\|_{C([0,1]; L^{p'}(-1,0; E))}) \right),$$

where  $L_{f_1}, L_{f_2}$  are the Lipschitz constant of  $f_1, f_2$ , respectively. Hence and since  $\|S(s)\|_{\mathcal{L}(C_0([0,1]))} \leq 1$  for all  $s > 0$ ,  $\phi$  satisfies (H $\phi$ ) with  $a(t) = L$ . Now, we prove that the assumption (H $\psi$ ) holds.

We prove that  $[0, t] \ni s \mapsto S(s)\psi \in C_0([0, 1])$  represents an operator  $R_{\psi, t}$  from  $\gamma(L^2(0, t), C_0[0, 1])$  defined as  $R_{\psi, t}f = \int_0^t S(s)\psi f(s)ds$ . For  $t = 1$  we have

$$(41) \quad R_{\psi, 1}f(\xi) = \left( \int_0^1 S(s)\psi f(s)ds \right)(\xi) = \int_0^1 e^{-\xi\mu} 1_{[s, 1]}(\xi) \psi(\xi - s) f(s) ds$$

$$(42) \quad = e^{-\xi\mu} \int_0^\xi \psi(\xi - s) f(s) ds$$

and then for all  $\xi \in [0, 1]$

$$(43) \quad |R_{\psi, 1}f(\xi)| \leq \|\psi\|_\infty \int_0^\xi f(t) dt.$$

Let  $h_0(\xi) = 1$ ,  $h_k(\xi) = 2^{\frac{1}{2}(n-1)} \left( 1_{(\frac{2j-2}{2^n}, \frac{2j-1}{2^n})}(\xi) - 1_{(\frac{2j-1}{2^n}, \frac{2j}{2^n})}(\xi) \right)$  for all  $\xi \in [0, 1]$  and  $k = 2^{n-1} + j - 1$  with  $n = 1, 2, \dots$ ,  $j = 1, 2, \dots, 2^{n-1}$  be the Haar basis on  $L^2(0, 1)$ . Then using (43) we obtain, for all  $\xi \in [0, 1]$  and  $k \geq 1$ ,

$$(44) \quad |R_{\psi, 1}h_k(\xi)| \leq \|\psi\|_\infty \int_0^1 h_k(t) dt \leq 1_{(\frac{2j-2}{2^n}, \frac{2j}{2^n})}(\xi) \|\psi\|_\infty 2^{\frac{1}{2}(n-1)} \frac{1}{2^n},$$

where  $k = 2^{n-1} + j - 1$ ,  $n = 1, 2, \dots, j = 1, \dots, 2^{n-1}$ .

Let  $\{\gamma_k : k = 0 \text{ or } k = 2^{n-1} + j - 1, n = 1, 2, \dots, j = 1, 2, \dots, 2^{n-1}\}$  be a Gaussian sequence, then for every  $\xi \in [0, 1]$  and all  $\beta > 1$  and sufficiently large  $1 < N < M$  such that  $N = 2^{n_N-1} + j_N - 1$  and  $N = 2^{n_M-1} + j_M - 1$  for some  $1 \leq j_N \leq 2^{n_N-1}$ ,  $1 \leq j_M \leq 2^{n_M-1}$  and  $n_N, n_M \geq 1$  we have, almost surely,

$$(45) \quad \begin{aligned} \sum_{k=N}^M |\gamma_k R_{\psi, 1}h_k(\xi)| &= \sum_{k=N}^M \sqrt{2\beta \log(k+1)} |R_{\psi, 1}h_k(\xi)| \\ &\leq \|\psi\|_\infty \sum_{n=n_N}^{m_M} \sum_{j=1}^{2^{n-1}} 1_{(\frac{2j-2}{2^n}, \frac{2j}{2^n})}(\xi) \sqrt{2\beta \log(j + 2^{n-1})} 2^{-\frac{1}{2}n - \frac{1}{2}} \\ &\leq \|\psi\|_\infty \sum_{n=n_N}^{m_M} \sqrt{2\beta \log(j' + 2^{n-1})} 2^{-\frac{1}{2}n - \frac{1}{2}} \\ &\leq \|\psi\|_\infty \sqrt{2\beta \log 2} \sum_{n=n_N}^{m_M} 2^{\frac{1}{2} + \frac{1}{4}n} 2^{-\frac{1}{2}n - \frac{1}{2}} \\ &= \|\psi\|_\infty \sqrt{2\beta \log 2} \sum_{n=n_N}^{m_M} \left( \frac{1}{\sqrt[4]{2}} \right)^n, \end{aligned}$$

where we use the following property of Gaussian sequences: for every  $\beta > 1$  the events  $|\gamma_k| \leq \sqrt{2\beta \log(k+1)}$  hold for all but finitely many  $k$  and the equalities:  $\log(2^{n-1} + j') \leq \log 2^n = n \log 2$  and  $\sqrt{n} \leq 2^{\frac{1}{2} + \frac{1}{4}n}$ . For all  $N = 2^{n-1} + j - 1$  with  $n = 1, 2, \dots$  and  $1 \leq j \leq 2^{n-1}$  let  $S_N(\xi) = \sum_{k=1}^N \gamma_k R_{\psi, 1}h_k(\xi)$ ,  $\xi \in [0, 1]$ . Hence the sequence  $(S_N)_{N \geq 1}$  converges to  $Y$ , almost surely, absolutely and uniformly for all  $\xi \in [0, 1]$ . Since each  $\xi \mapsto S_N(\xi)$  is continuous, it implies that the function



$\xi \mapsto Y(\xi)$  belongs to  $C[0, 1]$ . Moreover, in the same way as in (45) we obtain:

$$(46) \quad \mathbb{E} \|Y\|_\infty^2 = \mathbb{E} \sup_{\xi \in [0, 1]} \left| \sum_{k=1}^{\infty} \gamma_k R_{\psi, 1} h_k(\xi) \right|^2$$

$$(47) \quad \leq \|\psi\|_\infty^2 \left( \sum_{n=1}^{N-1} 2^{-n-1} + 2\beta \log 2 \left( \sum_{n=N}^{\infty} \left( \frac{1}{\sqrt[4]{2}} \right)^n \right)^2 \right) < \infty,$$

where  $N > 1$  is sufficiently large. By (45)-(46) and the Ito-Nisio theorem (see Proposition 2.11 in [8]) the sequence  $(S_n)_{n \geq 0}$  is convergence in  $L^2(\Omega; C([0, 1]))$  and a.s. to  $Y \in L^2(\Omega; C([0, 1]))$ . Therefore,  $R_{\psi, 1} \in \gamma(L^2(0, 1); C_0([0, 1]))$  and from Corollary 7.2 in [18] it follows that  $R_{\psi, t} \in \gamma(L^2(0, t); C_0([0, 1]))$  for all  $t > 0$ . Finally, by Proposition 5.6 we have the existence and uniqueness of weak solution to (37) in the spaces  $\mathbb{SL}_{\mathcal{F}}^{p, q}(0, t; E)$  for every  $q \geq 1$  and the weak solution satisfies

$$(48) \quad X(t, \xi) = 1_{[0, \infty)}(\xi - t) e^{-\xi \mu} x(\xi - t) + e^{-\xi \mu} \int_0^{t-\xi} \phi(X(s), X_s)(\xi - t + s) ds$$

$$(49) \quad + e^{-\xi \mu} \int_0^{t \wedge \xi} \psi(\xi - s) dW(s),$$

for all  $t > 0$  and  $\xi \in [0, 1]$ .

**5.2.2. Stochastic McKendrick equation with delay.** Let  $E = L^1(\mathcal{O})$ , where  $\mathcal{O} = (0, \infty)$ . Consider the following stochastic McKendrick equation with delay in  $L^1(\mathcal{O})$ :

$$(50) \quad \begin{cases} dy(t, \xi) = \left( -\frac{\partial y(t, \xi)}{\partial \xi} - \mu(\xi) y(t, \xi) \right) dt + \left[ \int_{t-1}^t \varphi(s - t, \xi) y(s, \xi) ds + f_1(y(t, \xi)) \right. \\ \quad \left. + \int_{t-1}^t k(s - t, \xi) f_2(y(s, \xi)) ds \right] dt + \psi(\xi) dW(t), \quad t \geq 0; \\ y(t, 0) = \int_0^\infty b(a) y(t, a) da; \\ y(0, \xi) = x_0(\xi), x_0 \in L^1(0, \infty) \quad y(\theta, \xi) = f_0(\theta, \xi), f_0 \in L^p(-1, 0; L^1(0, \infty)), \end{cases}$$

where  $p \geq 1$ ,  $\mu, b \in L^\infty(\mathcal{O})$ ,  $\varphi, k \in L^\infty(\mathcal{O}; L^{p'}(-1, 0))$  for some  $p' \in (1, \infty]$  such that  $\frac{1}{p} + \frac{1}{p'} = 1$ , and  $f_1, f_2 : \mathbb{R} \rightarrow \mathbb{R}$  are Lipschitz functions, and  $W$  is one-dimensional Brownian motion;  $\psi \in \mathcal{L}(\mathbb{R}, L^1(\mathcal{O}))$ ,  $\psi h(a) = h\sigma(a)$  for all  $h \in \mathbb{R}$  for some  $\sigma \in L^1(\mathcal{O})$  such that  $\text{supp } \sigma \subset [0, d]$  ( $d \in \mathbb{R}_+$ ) and  $\sigma \in L^2(0, d)$ . This equation can be used to model population dynamics (see [21]).

Let  $B$  be a linear operator on  $L^1(\mathcal{O})$  such that:

$$D(B) = \text{Ker}(K), \quad Bg = -\frac{d}{da} g - \mu g,$$

where  $K : W^{1,1}(\mathcal{O}) \rightarrow \mathbb{R}$ ,  $Kg = g(0) - \int_{\mathcal{O}} b(a) g(a) da$ . Then, by Theorem 2 in [3] it follows that  $B$  generates the McKendrick semigroup  $(S(t))_{t \geq 0}$ . Hence for all  $t \geq 0$

$$(51) \quad S(t)g(a) = e^{-\int_{a-t}^a \mu(r) dr} \tilde{g}(a - t), \quad a \geq 0,$$

where  $\tilde{g}(a) = g(a)$ ,  $\tilde{g}(-a) = g_2(a)$ ,  $a \geq 0$  and  $g_2$  belongs to the weighted Banach space

$$L_w^1(\mathcal{O}) = \left\{ g : \mathcal{O} \rightarrow \mathbb{R} : \|g\|_{L_w^1(\mathcal{O})} = \int_{\mathcal{O}} |g(a)| e^{-wa} da < \infty \right\} \text{ for some } w > \|b_\mu\|_\infty,$$

and satisfies, almost everywhere, the equation

$$(52) \quad g_2 = b_\mu \star g_2 + T_{\mu, b} g,$$

where

$$T_{\mu,b} : L^1(\mathcal{O}) \rightarrow L_w^1(\mathcal{O}), \quad T_{\mu,b}g(s) = \int_s^\infty e^{-\int_{a-s}^a \mu(r)dr} g(a-s)b(a)da,$$

$$b_\mu(s) = e^{-\int_0^s \mu(r)dr} b(s), \quad s \geq 0,$$

and  $\star$  denoted the convolution operation in  $L^1(\mathcal{O})$ . For every  $g \in L^1(\mathcal{O})$  the function  $\tilde{g} = (g, g_2) \in L^1(\mathcal{O}) \times L_w^1(\mathcal{O})$  defined by (52) is called  $(\mu, b)$ -extension of  $g$ .

Let us denote by  $\mathcal{E}_p = L^1(\mathcal{O}) \times L^p(-1, 0; L^1(\mathcal{O}))$  the state space for the delay equation (50). Let  $\phi : \mathcal{E}_p \rightarrow L^1(\mathcal{O})$  be give by 39. It is easy to show that  $\phi$  is Lipschitz-continuous with the Lipschitz constant

$$L = 2^{\frac{1}{p}} \left( L_{f_1} \vee (L_{f_2} \|k\|_{L^\infty(\mathcal{O}; L^{p'}(-1, 0; E))} + \|\varphi\|_{L^\infty(\mathcal{O}; L^{p'}(-1, 0; E))}) \right),$$

where  $L_{f_1}, L_{f_2}$  are the Lipschitz constant of  $f_1, f_2$ , respectively. Hence  $\phi$  satisfies  $(H\phi)$  with  $a(t) = LS(t)$ . We show in Proposition 5.7 that the assumptions  $(H\psi)$  and (32)-(33) holds. Therefore, we can rewrite (37) in the form (29) and apply Proposition 5.6 to prove the existence, uniqueness and continuity of weak solution to (50).

**Proposition 5.7.** *Consider (50). Then, the operator  $\psi \in \mathcal{L}(\mathbb{R}, L^1(\mathcal{O}))$  defined by  $\psi h(a) = h\sigma(a)$  for all  $h \in \mathbb{R}$  and for some  $\sigma \in L^1(\mathcal{O})$  such that  $\text{supp } \sigma \subset [0, d]$  ( $d \in \mathbb{R}_+$ ) and  $\sigma \in L^2(0, d)$  satisfies  $(H\psi)$  and (32)-(33).*

*Proof.* The  $\gamma$ -Fubini isomorphism (see Proposition 2.6 in [16]) between the Banach spaces  $L^1(\mathcal{O}; (L^2(0, t))^*)$  and  $\gamma(L^2(0, t); L^1(\mathcal{O}))$  implies that to prove condition  $(H\psi)$  it is enough to find  $t > 0$  such that

$$(53) \quad \sup_{\mathcal{O} \parallel f \parallel_{L^2(0, t)} \leq 1} \left| \left( \int_0^t S(s)\psi f(s)ds \right) (a) \right| da < \infty.$$

We take  $t = d$ , then for a.e.  $a \geq 0$  by the Cauchy-Schwarz inequality it follows that

$$\left| \left( \int_0^d S(s)\psi f(s)ds \right) (a) \right| \leq \|f\|_{L^2(0, d)} \left( \int_0^d e^{-2\int_{a-s}^a \mu(r)dr} \tilde{\sigma}^2(a-s)ds \right)^{\frac{1}{2}}.$$

Hence

$$(54) \quad \int_{\mathcal{O}} \sup_{\parallel f \parallel_{L^2(0, d)} \leq 1} \left| \int_0^d S(s)\psi f(s)ds \right| (a) da \leq \int_{\mathcal{O}} \left( \int_0^d e^{-2\int_{a-s}^a \mu(r)dr} \tilde{\sigma}^2(a-s)ds \right)^{\frac{1}{2}} da$$

$$\leq \int_0^d \left( \int_0^d e^{-2\int_{a-s}^a \mu(r)dr} \tilde{\sigma}^2(a-s)ds \right)^{\frac{1}{2}} da$$

$$+ \int_d^\infty \left( \int_0^d e^{-2\int_{a-s}^a \mu(r)dr} \sigma^2(a-s)ds \right)^{\frac{1}{2}} da.$$

Since  $\sigma \in L^2(0, d)$ , we can estimate the second integral on right hand side of (54) as follows

$$\begin{aligned} \int_d^\infty \left( \int_0^d e^{-2 \int_{a-s}^a \mu(r) dr} \sigma^2(a-s) ds \right)^{\frac{1}{2}} da &= \int_d^\infty \left( \int_{a-d}^a e^{-2 \int_s^a \mu(r) dr} \sigma^2(s) ds \right)^{\frac{1}{2}} da \\ &= \int_d^{2d} \left( \int_{a-d}^d e^{-2 \int_s^a \mu(r) dr} \sigma^2(s) ds \right)^{\frac{1}{2}} da \leq d \|\sigma\|_{L^2(0, d)}. \end{aligned}$$

For the first integral on right hand side of (54) we obtain

$$\begin{aligned} \int_0^d \left( \int_0^d e^{-2 \int_{a-s}^a \mu(r) dr} \tilde{\sigma}^2(a-s) ds \right)^{\frac{1}{2}} da &\leq \int_0^d \left( \int_0^a e^{-2 \int_{a-s}^a \mu(r) dr} \sigma^2(a-s) ds \right)^{\frac{1}{2}} da \\ &\quad + \int_0^d \left( \int_a^d e^{-2 \int_0^a \mu(r) dr} \sigma_2^2(s-a) ds \right)^{\frac{1}{2}} da \\ &\leq d \|\sigma\|_{L^2(0, d)} + d \|\sigma_2\|_{L^2(0, d)}, \end{aligned}$$

if  $\sigma_2$  is square integrable on  $(0, d)$ . We show that  $\sigma_2 \in L^2(0, d)$ . Recall that  $\sigma_2 \in L_w^1(\mathcal{O})$  for  $w > \|b_\mu\|_\infty$  is the solution to (see (52)):

$$(55) \quad \sigma_2 = b_\mu \star \sigma_2 + T_{\mu, b} \sigma.$$

We denote the restriction of  $T_{\mu, b}$  to  $L^2(0, d)$  by  $T_{\mu, b, 2}$ . Since for all  $s \in [0, t]$  and  $g \in L^2(0, d)$

$$(56) \quad \begin{aligned} T_{\mu, b, 2} g(s) &= \int_s^\infty e^{-\int_{a-s}^a \mu(r) dr} b(a) g(a-s) da = \int_0^d e^{-\int_a^{a+s} \mu(r) dr} b(a+s) g(a) da, \\ \|T_{\mu, b, 2} g\|_{L_w^2(0, d)} &\leq \sqrt{d} \|b\|_\infty \|g\|_{L^2(0, d)}, \end{aligned}$$

hence  $T_{\mu, b, 2} \in \mathcal{L}(L^2(0, d), L_w^2(0, d))$ . Moreover, by the Young inequality for convolutions it follows that for all  $g \in L_w^2(0, d)$  we have

$$\|b_\mu \star g\|_{L_w^2(0, d)} \leq \|b_\mu\|_{L_w^1(0, d)} \|g\|_{L_w^2(0, d)} \leq \frac{\|b_\mu\|_\infty}{w} \|g\|_{L_w^2(0, d)},$$

Hence for  $w > \|b_\mu\|_\infty$  the mapping  $L_w^2(0, d) \ni g \mapsto b_\mu \star g + T_{\mu, b, 2} \sigma \in L_w^2(0, d)$  is a strict contraction, thus by the Banach fixed-point theorem there exists a unique  $\sigma_2 \in L_w^2(0, d)$  solution to (55). It is clear that  $L_w^2(0, d)$  and  $L^2(0, d)$  are isomorphic, thus  $\sigma_2 \in L^2(0, d)$ . Therefore, the proof of (53) is complete.

Let  $\alpha \in (\frac{1}{q\sqrt{p}}, \frac{1}{2})$  and  $t > 0$ . Now we prove that

$$\sup_{s \in [0, t]} \|u \mapsto (s-u)^{-\alpha} S(s-u) \psi\|_{\gamma(L^2(0, s), L^1(\mathcal{O}))} < \infty.$$

Fix  $s \in [0, t]$ . Notice that by  $\gamma$ -Fubini isomorphism we have

$$\begin{aligned} &\|u \mapsto (s-u)^{-\alpha} S(s-u) \psi\|_{\gamma(L^2(0, s), L^1(\mathcal{O}))} \\ &\leq C_\gamma \int_{\mathcal{O}} \sup_{\|f\|_{L^2(0, s)} \leq 1} \left| \left( \int_0^s (s-u)^{-\alpha} S(s-u) \psi f(u) du \right) (a) \right| da, \end{aligned}$$

for some constant  $C_\gamma > 0$ . By the Cauchy-Schwarz inequality

$$\begin{aligned}
 (57) \quad & \int_{\mathcal{O}} \sup_{\|f\|_{L^2(0,s)} \leq 1} \left| \left( \int_0^s (s-u)^{-\alpha} S(s-u) \psi f(u) du \right) (a) \right| da \\
 (58) \quad & \leq \int_0^s \left( \int_0^s u^{-2\alpha} e^{-2 \int_{a-u}^a \mu(r) dr} \tilde{\sigma}^2(a-u) ds \right)^{\frac{1}{2}} da \\
 & \quad + \int_s^\infty \left( \int_0^s u^{-2\alpha} e^{-2 \int_{a-u}^a \mu(r) dr} \sigma^2(a-u) ds \right)^{\frac{1}{2}} da.
 \end{aligned}$$

Using the assumption  $\sigma \in L^2(0, d)$  and then the Cauchy Schwarz inequality and Fubini's theorem we can estimate the second integral on the right hand side of (57) as follows

$$\begin{aligned}
 & \int_s^\infty \left( \int_0^s u^{-2\alpha} e^{-2 \int_{a-u}^a \mu(r) dr} \sigma^2(a-u) ds \right)^{\frac{1}{2}} da \\
 & = \int_s^{s+d} \left( \int_{a-s}^{d \wedge a} (a-u)^{-2\alpha} e^{-2 \int_u^a \mu(r) dr} \sigma^2(u) ds \right)^{\frac{1}{2}} da \\
 & \leq \sqrt{d} \left( \int_s^{s+d} \int_{a-s}^{d \wedge a} (a-u)^{-2\alpha} \sigma^2(u) ds da \right)^{\frac{1}{2}} \\
 & = \sqrt{d} \left( \int_0^d \sigma^2(u) \int_{s \vee u}^{u+s} (a-u)^{-2\alpha} dadu \right)^{\frac{1}{2}} \\
 & \leq \sqrt{d \frac{s^{1-2\alpha} - 1}{1-2\alpha}} \|\sigma\|_{L^2(0,d)}.
 \end{aligned}$$

We decompose the first term on the right hand side of (57) as

$$\begin{aligned}
 & \int_0^s \left( \int_0^s u^{-2\alpha} e^{-2 \int_{a-u}^a \mu(r) dr} \tilde{\sigma}^2(a-u) ds \right)^{\frac{1}{2}} da \\
 & = \int_0^s \left( \int_0^a u^{-2\alpha} e^{-2 \int_{a-u}^a \mu(r) dr} \sigma^2(a-u) ds \right)^{\frac{1}{2}} da \\
 & \quad + \int_0^s \left( \int_a^s u^{-2\alpha} e^{-2 \int_{a-u}^a \mu(r) dr} \sigma_2^2(a-u) ds \right)^{\frac{1}{2}} da.
 \end{aligned}$$

The Cauchy-Schwarz inequality and Fubini's theorem gives

$$\begin{aligned}
 \int_0^s \left( \int_0^a u^{-2\alpha} e^{-2 \int_{a-u}^a \mu(r) dr} \sigma^2(a-u) ds \right)^{\frac{1}{2}} da & \leq \sqrt{s} \left( \int_0^s u^{-2\alpha} \int_0^{s-u} \sigma^2(a) dadu \right)^{\frac{1}{2}} \\
 & \leq \sqrt{\frac{s(s^{1-2\alpha} - 1)}{1-2\alpha}} \|\sigma\|_{L^2(0,d)}.
 \end{aligned}$$

and

$$\begin{aligned} \int_0^s \left( \int_a^s u^{-2\alpha} e^{-2 \int_{a-u}^a \mu(r) dr} \sigma_2^2(a-u) ds \right)^{\frac{1}{2}} da &\leq \sqrt{s} \left( \int_0^s u^{-2\alpha} \int_0^u \sigma_2^2(a) da du \right)^{\frac{1}{2}} \\ &\leq \sqrt{\frac{s(s^{1-2\alpha}-1)}{1-2\alpha}} \|\sigma_2\|_{L^2(0,s)}. \end{aligned}$$

Similarly as in the first part of the proof we obtain that  $\sigma_2 \in L^2(0, s)$  for all  $s > 0$ . Therefore,

$$\begin{aligned} \sup_{s \in [0, t]} \|u \mapsto (s-u)^{-\alpha} S(s-u)\psi\|_{\gamma(L^2(0,s), L^1(\mathcal{O}))} \\ \leq C_\gamma \sqrt{(d \vee t) \frac{t^{1-2\alpha}-1}{1-2\alpha}} (2\|\sigma\|_{L^2(0,d)} + \|\sigma_2\|_{L^2(0,t)}) < \infty. \end{aligned}$$

□

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